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NOTE

Extendibility of Rational Matrices

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A Characterization of extendibility of rational matrices is presented in terms of elementary properties. As a tool we give a solvability condition for a system of linear diophantine equations, which is of independent interest. © 1997 Academic Press

The property of extendibility of rational matrices was introduced by Sivakumar [3] in his investigation of linear independence of integer translates of exponential box splines with rational directions. This property was subsequently extended and refined by Ron [2]. The purpose of this note is to provide a characterization of extendibility in terms of elementary properties.

DEFINITION. Let $Y \subset \mathbf{Q}^s$ be a linearly independent set of $1 \leqslant k \leqslant s$ vectors. We say that Y is extendible if there is a matrix $X_{s \times s}$ with an integral inverse whose first k columns constitute Y. For an arbitrary $s \times n$ matrix Ξ, we say that Ξ is fully extendible if every linearly independent subset Y of Ξ is extendible.

Note that any $s \times n$ rational matrix can be written as $(1/P)\,\Xi$ with $P \in \mathbf{N}$ and $\Xi \in \mathbf{Z}^{s \times n}$, which is crucial in our investigation of box splines with rational directions [5]. So in what follows we always take such a form for a rational matrix. For an $l \times m$ integer matrix A we also think of it as the multiset of its column vectors and denote $\# A$ as its cardinality. Also define $d_{A,r}$ as the greatest common divisor of all $r \times r$ minors of A. Set $d_{A,0} = 1$. Then our main result can be stated as follows.

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Theorem 1. *Let* $\Xi \in \mathbf{Z}^{s \times n}$ *and* $P \in \mathbf{N}$. *Then* $(1/P)\Xi$ *is fully extendible if and only if for every linearly independent maximal subset* $Y \subset \Xi$, *i.e.,* $\# Y = \operatorname{rank} Y = \operatorname{rank} \Xi$, *there holds*

$$\frac{d_{Y, \# Y}}{d_{Y, \# Y - 1}} \Big| P. \tag{1}$$

Remark. In the bivariate case a characterization of extendibility was obtained in [3, Prop. 3.6].

To prove Theorem 1, we need to extend two results of Jia [1, Corollary 3.3] and the author [4, Lemma 2] on solvability of a system of linear diophantine equations, which is of independent interest.

Theorem 2. *Let* $A \in \mathbf{Z}^{l \times m}$ *be an integer matrix of full row rank and* $P \in \mathbf{N}$. *Then the following system of linear diophantine equations*

$$Ay = Pb \tag{2}$$

has integer solutions for any $b \in \mathbf{Z}^l$ *if and only if*

$$\frac{d_{A,l}}{d_{A,l-1}} \Big| P. \tag{3}$$

Proof of Theorem 2. We use the method of Jia [1].

By [1, Theorem 3.2] the sufficiency is trivial.

To prove the necessity, we let b be e_j^l, the jth column of the $l \times l$ identity matrix I_l, then the system of linear diophantine equations (2) has integer solutions, which implies by [1, Theorem 3.2] $d_{A,l} = d_{[A, Pe_j^l], l}$, $1 \leqslant j \leqslant l$. Hence $d_{A,l} \mid P d_{A,l-1}$. Note that $d_{A,l-1} \mid d_{A,l}$. The conclusion (3) is obtained. The proof of Theorem 2 is complete.

Once Theorem 2 holds, Theorem 1 follows.

Proof of Theorem 1. It is easily seen that $(1/P)\Xi$ is fully extendible if every linearly independent maximal subset is extendible.

Let $Y := \{y_1, ..., y_l\} \subset \Xi$ satisfy $l = \operatorname{rank} Y = \operatorname{rank} \Xi$. We state that $(1/P)\, Y$ is extendible if and only if there exists a basis $Z := \{z_1, ..., z_l\} \subset \mathbf{Z}^s$ dual to $(1/P)\, Y$, i.e.,

$$\frac{1}{P} Y^T Z = I_l. \tag{4}$$

The necessity of this statement follows directly from the definition.

To see the sufficiency, choose $\{\tilde{z}_j : l+1 \leqslant j \leqslant s\} \subset \mathbf{Z}^s$ such that $\{z_j : 1 \leqslant j \leqslant l\} \cup \{\tilde{z}_j : l+1 \leqslant j \leqslant s\}$ are linearly independent. Then we define for $l+1 \leqslant j \leqslant s$,

$$z_j = P\tilde{z}_j - \sum_{k=1}^{l} (y_k^T \tilde{z}_j) z_k \in \mathbf{Z}^s. \tag{5}$$

Trivially, $\{z_j : 1 \leqslant j \leqslant s\}$ are linearly independent, and the first l columns of $([z_1, ..., z_s]^T)^{-1} \in \mathbf{Q}^{s \times s}$ constitute $(1/P)\, Y$, since for $1 \leqslant i \leqslant l$, $[z_1, ..., z_s]^T (1/P)\, y_i = ((1/P)\, y_i^T [z_1, ..., z_s])^T = e_i^s$.

Thus the matrix $(1/P)\, Y$ is extendible if and only if for any $b \in \mathbf{Z}^l$, the following system of linear diophantine equations

$$Y^T y = Pb$$

has integer solutions, which is equivalent to (1) by Theorem 2.

The proof of Theorem 1 is complete.

In the proof of Theorem 1, we have in fact shown the following more general result.

THEOREM 3. *Let $P \in \mathbf{N}$ and $Y \subset \mathbf{Z}^s$ be a linearly independent set. Then $(1/P)\, Y$ is extendible if and only if*

$$\left. \frac{d_{Y, \# Y}}{d_{Y, \# Y - 1}} \right| P.$$

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